# Quiz 4 Solutions

1. Let $f(x)$ be a function with Taylor series given by

$$\sum_{n=0}^{\infty} c_n (x-2)^n$$

and suppose the radius of convergence is $R=5$.

(a) Where is the Taylor series centered and what can you say about the interval of convergence?

The Taylor series is centered at $a=2$. The interval of convergence is one of the following four options

$$[-3,7],\ [-3,7),\ (-3,7],\ (-3,7)$$

(b) Recall that the coefficients of a Taylor series are given by

$$c_n = \frac{f^{(n)}(a)}{n!}$$

If $c_{13}=104$, what is $f^{(13)}(2)$?

$$104 = c_{13} = \frac{f^{(13)}(2)}{13!}$$

$$f^{(13)}(2) = 104 \cdot 13!$$

(c) If possible, write a series for $f(0)$. If it is not possible, explain why not.

$$f(0) = \sum_{n=0}^{\infty} c_n (-2)^n$$

(d) If possible, write a series for $f(8)$. If it is not possible, explain why not. Since 8 is outside the interval of convergence, we cannot express $f(8)$ as a series with the given information.

2. Write a Maclaurin series for $f(x) = x\sin(2x)$. What is the interval of convergence?

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad \text{IoC: } \mathbb{R}$$

$$\sin(2x) = \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{(2n+1)!} \quad \text{IoC: } \mathbb{R}$$

$$\boxed{x\sin(2x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+2}}{(2n+1)!} \quad \text{IoC: } \mathbb{R}}$$

3. Write a Maclaurin series for $f(x) = \frac{x^4}{1+x}$.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{IoC: } |x| < 1$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n \quad \text{IoC: } |-x| < 1 \text{ i.e., } |x| < 1$$

$$\boxed{\frac{x^4}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^{n+4} \quad \text{IoC: } |x| < 1 \text{ i.e., } (-1, 1)}$$

4. Express $\int \frac{e^x}{x}\,dx$ as an infinite series.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \text{IoC}: \mathbb{R}$$

$$\frac{e^x}{x} = \sum_{n=0}^{\infty} \frac{x^{n-1}}{n!} = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!} \qquad \text{IoC}: \mathbb{R}$$

$$\boxed{\int \frac{e^x}{x}\,dx = \left(\sum_{n=1}^{\infty} \frac{x^n}{n \cdot n!}\right) + \ln|x| + C \quad \text{IoC}: \mathbb{R}}$$

* This is an infinite series but not a power series. I meant to subtract of the pesky $\frac{1}{x}$ term. My bad!

5. Use series to approximate $\int_0^1 \sqrt{1+x^4}\,dx$ correct to two decimal places.

* This question has a typo. The integral needs to have an upper limit less than 1 to be done the nice way I intended. If you attempted this question, you got full credit. See Section 8.7, example 11 for a worked out example of this method.

6. Use series to evaluate the following limit:

$$\lim_{x \to 0} \frac{\sin x - x}{x^3}$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad \text{IoC: } \mathbb{R}$$

$$\sin x - x = \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad \text{IoC: } \mathbb{R}$$

$$\frac{\sin x - x}{x^3} = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-2}}{(2n+1)!} \qquad \text{IoC: } \mathbb{R}$$

$$= \frac{-1}{3!} + \frac{x^2}{5!} - \frac{x^4}{7!} + \cdots$$

$$\lim_{x \to 0} \frac{\sin x - x}{x^3} = \lim_{x \to 0} \left( \frac{-1}{3!} + \frac{x^2}{5!} - \frac{x^4}{7!} + \cdots \right)$$

$$\boxed{= -\frac{1}{3!}}$$

7. Find the sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{6^{2n}(2n)!}$.

$$\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{6^{2n}(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{6}\right)^{2n}}{(2n)!} = \cos\left(\frac{\pi}{6}\right) = \boxed{\frac{\sqrt{3}}{2}}$$

$\boxed{B}$

8. Give the radius and interval of convergence for the power series

$$\sum_{n=1}^{\infty} \frac{5^n}{2n}(2x-1)^{n+1}$$

**Radius:**

$$\lim_{n\to\infty} \left| \frac{\frac{5^{n+1}}{2(n+1)}(2x-1)^{n+2}}{\frac{5^n}{2n}(2x-1)^{n+1}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{5(2x-1)n}{n+1} \right|$$

$$= 5|2x-1| < 1$$

$$|2x-1| < \frac{1}{5}$$

$$-\frac{1}{5} < 2x-1 < \frac{1}{5}$$

$$\frac{4}{5} < 2x < \frac{6}{5}$$

$$\frac{2}{5} < x < \frac{3}{5}$$

$$\boxed{R = \frac{1}{10}}$$

End pts: $x = \frac{2}{5}$ : $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{2n}\left(2\cdot\frac{2}{5}-1\right)^{n+1} = \sum_{n=1}^{\infty} \frac{5^n}{2n}\left(\frac{-1}{5}\right)^{n+1}$

Alternating harmonic-ish $\rightarrow$ $\displaystyle = \frac{-1}{5}\sum_{n=1}^{\infty} \frac{(-1)^n}{2n}$ converges

$x = \frac{3}{5}$: $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{2n}\left(2\cdot\frac{3}{5}-1\right)^{n+1} = \sum_{n=1}^{\infty} \frac{5^n}{2n}\left(\frac{1}{5}\right)^{n+1}$ $\leftarrow$ harmonic-ish

$\displaystyle = \frac{1}{5}\sum_{n=1}^{\infty} \frac{1}{2n}$ $\leftarrow$ DIVERGES

$$\boxed{\text{Interval of Convergence: } \left[\frac{2}{5}, \frac{3}{5}\right)}$$

9. Which function is a solution to the differential equation

$$y'' + y = 2\cos(x) \ ?$$

(a) $y = \cos(2x)$

$y' = -2\sin(2x)$

$y'' = -4\cos(2x)$

$$y'' + y = -4\cos(2x) + \cos(2x) = -3\cos(2x) \neq 2\cos(x)$$

Sad day :(

(b) $y = \sin(x^2)$

$y' = 2x\cos(x^2)$

$y'' = -4x^2\sin(x^2) + 2\cos(x^2)$

$$y'' + y = -4x^2\sin(x^2) + 2\cos(x^2) + \sin(x^2) \neq 2\cos(x)$$

:(

(c) $y = x\sin(x)$

$y' = \sin(x) + x\cos(x)$

$y'' = \cos(x) + \cos(x) - x\sin(x)$

$$y'' + y = 2\cos(x) - x\sin(x) + x\sin(x) = 2\cos(x) \checkmark$$

WOO!

$\boxed{C}$

10. Which function is a solution to the differential equation

$$y = \left(\frac{y''}{6}\right)^3 ?$$

(a) $y = (x-2)^3$

$y' = 3(x-2)^2$

$y'' = 6(x-2)$

$$\left(\frac{y''}{6}\right)^3 = \left(\frac{6(x-2)}{6}\right)^3 = (x-2)^3 = y \checkmark$$

Yay! first try!

$\boxed{A}$

11. Solve the differential equation $y' = \frac{xy^3}{\sqrt{1+x^2}}$ given the initial condition $y(0) = -1$.

$$\frac{dy}{dx} = \frac{xy^3}{\sqrt{1+x^2}}$$

$$y^{-3}\frac{dy}{dx} = \frac{x}{\sqrt{1+x^2}}$$

$$\int y^{-3}\,dy = \int \frac{x}{\sqrt{1+x^2}}\,dx$$

$$\frac{y^{-2}}{-2} = \sqrt{1+x^2} + C$$

$$\frac{-1}{2y^2} = \sqrt{1+x^2} + C$$

$y(0) = -1$:

$$\frac{-1}{2} = 1 + C$$

$$C = -\frac{3}{2}$$

$$\frac{-1}{2y^2} = \sqrt{1+x^2} - \frac{3}{2}$$

$$-2y^2 = \frac{1}{\sqrt{1+x^2} - \frac{3}{2}}$$

$$y^2 = \frac{1}{2\sqrt{1+x^2} - 3}$$

$$\boxed{y = \frac{-1}{\sqrt{2\sqrt{1+x^2} - 3}}}$$

12.
- A tank contains 1000L of pure water.
- Brine containing 0.5 kg of salt per Liter enters the tank at a ratte of 5 L/min.
- Brine that contains 0.4 kg of salt per liter enters the tank at a rate of 10 L/min.
- Tank is kept thoroughly mixed
- Brine drains from the tank at a rate of 15 L/min
- How much salt is in the tank after $t$ minutes?
- How much salt is in the tank after 1 hour?

Let $C(t)$ be the concentration of salt in the tank at time $t$ and let $S(t)$ be the amount of salt in the tank at time $t$.

$C(0) = 0, \ S(0) = 0 \qquad C(t) = \frac{S(t)}{1000}$

Flow of salt into the tank: $0.5 \cdot 5 + 0.4 \cdot 10 = 6.5$ kg/min

Flow of salt out of the tank: $C(t) \cdot 15 = \frac{15\,S(t)}{1000}$

Net change of salt in the tank:

$$\frac{dS}{dt} = 6.5 - \frac{15\,S(t)}{1000} = \frac{650 - 15\,S(t)}{1000}$$

$$\int \frac{1}{650 - 15\,S(t)} \frac{dS}{dt}\,dt = \int \frac{1}{1000}\,dt$$

$$\ln|650 - 15\,S(t)| = \frac{1}{1000}t + C$$

$$650 - 15\,S(t) = C e^{\frac{1}{1000}t}$$

$$650 = C$$

$$-15\,S(t) = 650 e^{\frac{1}{1000}t} - 650$$

$$\boxed{S(t) = \frac{650}{15}\left(1 - e^{\frac{1}{1000}t}\right)}$$

$$\boxed{S(60) = \frac{650}{15}\left(1 - e^{60/1000}\right)}$$